Stability of plane Couette suspension flow with nonuniform particle concentration profile

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Abstract

The linear stability of plane Couette suspension flow in the presence of particle concentration gradients is considered. The dispersed flow is described within the framework of a modified two-fluid model which takes into account a finite volume fraction of the inclusions and particle velocity slip. The system of linearized governing equations is reduced to a fourth-order complex ordinary differential equation. The resulting eigenvalue problem is solved by means of the orthonormalization method. It is obtained that, in the presence of particles, Couette flow is unstable even at very low Reynolds numbers. The instability strongly depends on the particle concentration profile in the main flow.

Introduction

It is well known that plane Couette flow of an incompressible viscous fluid is stable with respect to small disturbances. As it is shown in [1], the injection of spherical particles (rigid or liquid) of low volume fraction can destabilize the flow. The two-phase flow was described in the framework of a dusty-gas model [2, 3], in which the particle volume fraction was neglected. It was shown that plane Couette flow of the heterogeneous medium with the non-uniform (Gaussian) concentration profile of the dispersed phase is unstable. The instability triggers if the maximum of the particle mass concentration exceeds a certain threshold value which depends on the steepness of the concentration profile of the inclusions. The increment magnitude of the growing waves corresponds to one of the Tollmien-Schlichting waves. The unstable disturbances exist for sufficiently high values of the Reynolds number, ranging from hundreds to dozens of millions. The dusty-gas model used in [1] for the description of the dispersed flow is applicable for particle-laden gas flows, when the ratio of the particle substance density to the carrier fluid density is large. For studying the stability of a suspension flows, in which the carrier-fluid density is of order of the particle substance density and the volume fraction of inclusions (rigid or liquid) is finite, a more complex model of heterogeneous medium is required. In what follows, we evaluate the effect of the finite particle volume fraction on the stability of dispersed flows by investigating the stability of plane Couette flow with a non-uniform spatial distribution of inclusions.

Problem formulation

We consider the stability of plane Couette suspension flow in the presence of particle concentration gradients. The flow is maintained by two plates moving relative to each other with the velocity $2U$, the distance between the plates is $2L$. The x-axis of the Oxy coordinate system is directed streamwise, while y-axis is perpendicular to the plates. The dispersed flow is described within the framework of the two-fluid model [2, 3] with modifications taking into account a finite particle volume fraction via corrections to the effective viscosity of the suspension and to the interphase force. The governing equations for the suspension flow in the non-dimensional form are as follows [4]:

$$\mathrm{div}((1-C)\mathbf{v}+C\mathbf{v}_s)=0, \quad (1-C)\frac{d\mathbf{v}}{dt}+\eta C\frac{d_s\mathbf{v}_s}{dt}=-\nabla p+\nabla_j\tau^{ij}\mathbf{e}_i \tag{1}$$

$$\frac{\partial C}{\partial t}+\mathrm{div}\,(C\mathbf{v}_s)=0, \quad \frac{d_s\mathbf{v}_s}{dt}=\beta(\mathbf{v}-\mathbf{v_s})\left(1+\sqrt{\frac{9C}{2}}\right) \tag{2}$$

$$\tau^{ij}=2\mu(C)\left(e^{ij}-\frac{1}{3}\delta^{ij}\,\mathrm{div}\,\mathbf{v}\right),\ e^{ij}=\frac{1}{2}\left(\nabla^i v^j+\nabla^j v^i\right),\ \mu=1+\frac{5}{2}C \tag{3}$$

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$$\mathrm{Re} = \frac{U_0 L \rho^0}{\mu_0}, \quad \beta = \frac{6\pi\sigma\mu_0 L}{m U_0}, \quad \eta = \frac{\rho_s^0}{\rho^0}$$

Here, parameters with subscript 's' correspond to the particulate phase, Re is the Reynolds number, β is the particle inertia parameter (inversed Stokes number), and η is the particle-to-fluid substance density ratio. The interphase force (right-hand side of the second of Eq.(2)) corresponds to the Stokes force with the Brinkman correction [5]. Effective viscosity of the suspension (see Eq.(3)) depends linearly on the particle volume fraction according to the Einstein formula [6]. We note that these corrections are of different orders with respect to C, so that for sufficiently low particle volume fractions, the correction to the viscosity is much lower than that to the Stokes force. However, it is the dependence of the effective viscosity of the suspension on the particle volume fraction that qualitatively changes the stability of the flow, so that growing disturbances of non-Tollmien-Schlichting type are triggered [4]. Therefore, we expect a dramatic decrease in the stability of suspension plane Couette flow in the presence of particle concentration gradients.

In the main flow, the particle concentration profile $C_0(y)$ is specified analytically as follows:

$$C_0(y) = \begin{cases} C_1 + (C_2 - C_1)\exp\left(-(y-y_0)^2/\varepsilon^2\right), & y \in [y_0, 1] \\ C_2, & y \in [-y_0, y_0] \\ C_1 + (C_2 - C_1)\exp\left(-(y+y_0)^2/\varepsilon^2\right), & y \in [-1, -y_0] \end{cases} \tag{4}$$

Here, the parameters C_1, C_2 ($C_1 \ll C_2$) correspond to the particle volume fraction in the vicinity of the walls and in the middle plane of the flow respectively, ε corresponds to the particle concentration steepness, while y_0 signifies the width of the particle-laden zone.

There is no particle velocity slip in the main flow. The velocity profile of both the carrier fluid and the particles differs from simple linear profile $U = y$ due to the nonuniformity of the effective-viscosity profile. Due to the total momentum conservation equation (1), the velocity of the suspension is described by the ordinary differential equation with no-slip boundary conditions on the walls.

We consider disturbances in the form of traveling waves

$$Q(x, y, t) = q(y)\exp\{ikx - \omega t)\},$$

where k is a real wave number and ω is a complex wave frequency. Based on the continuity equation for the suspension (1), we introduce the perturbation stream function of magnitude ψ:

$$(1 - C_0)u + C_0 u_s = \psi', \quad (1 - C_0)v + C_0 v_s = -ik\psi$$

The linearized system of governing equations is reduced to the fourth-order ordinary differential equation for the magnitude of the stream-function perturbation $\psi(y)$:

$$C_0'[ik(U-c)(\eta u_s - u) + U'(\eta v_s - v)] + ik(U-c)[u_*' - ikv_*] +$$
$$+ U'(iku_* + v_*') + v_* U'' = \frac{1}{\mathrm{Re}}\left\{\frac{5}{2}\left[(C'' + k^2 C)U' + 2C'U'' + CU''' + 2C_0'(u'' - k^2 u)\right] +\right.$$
$$\left. + \frac{5}{2}\left[C_0''(u' + ikv)\right] + \left(1 + \frac{5}{2}C_0\right)\left[-k^2 u' + u''' + ik^3 v - ikv''\right]\right\}$$

Here, all parameters u, v, u_s, v_s, u_*, v_*, C are linear functions of ψ and its derivatives. We do not present the exact expressions for these functions since they are too cumbersome. The no-slip boundary conditions are satisfied on the rigid walls:

$$\psi(\pm 1) = 0, \quad \psi'(\pm 1) = 0$$

The eigenvalue problem is solved by means of the orthonormalization method [7].

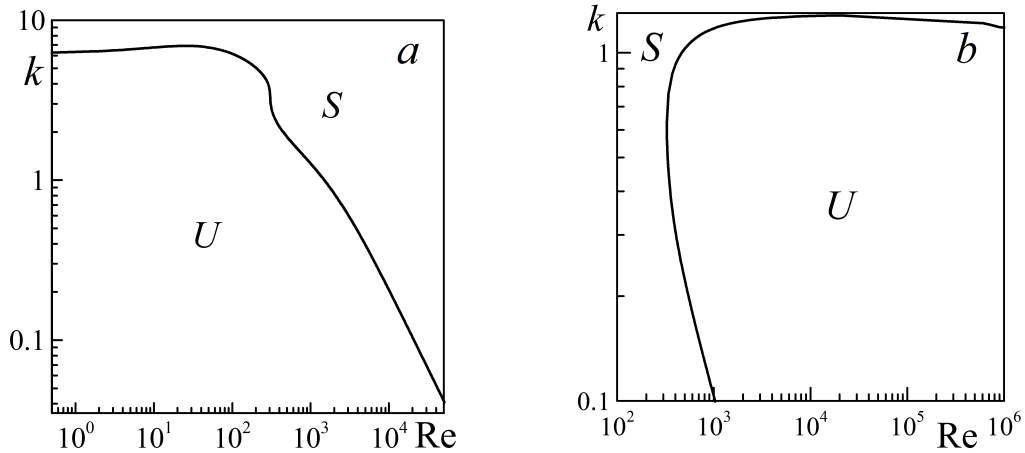

Figure 1. Neutral curves in the (Re, k) plane for the suspension flow for the parameters $C_1 = 0.001$, $C_2 = 0.1$, $y_0 = 0.5$, $\varepsilon = 0.1$, $\beta = 100$, $\eta = 1$ (*a*) and typical neutral curve for the dusty-gas flow (*b*) [1]. S and U indicate the stability and instability regions respectively

Results and Discussion

A parametric study of the eigenvalues based on the numerical calculations is performed. It is obtained that, due to the viscosity stratification, Couette suspension flow in the presence of particles distributed according to (4) with $y_0 > 0.02$ is unstable even at very low Reynolds numbers (Fig. 1*a*). In contrast, growing Tollmien-Schlichting waves are triggered only at sufficiently high Reynolds numbers [1] (Fig. 1*b*). The instability due to the stratification of finite particle volume fraction vanishes at high Reynolds numbers.

In contrast to the instability found in [1], the disturbances with high growth increments exist regardless of the finite value of the maximum particle mass concentration (Fig. 2). The maximum of the wave growth increment corresponds to the wave number of order unity (Fig. 2*a*). Studying the dependence of the maximum growth increment on the Reynolds number, one can see that the instability is most pronounced at $R \sim 100$ (Fig. 2*b*).

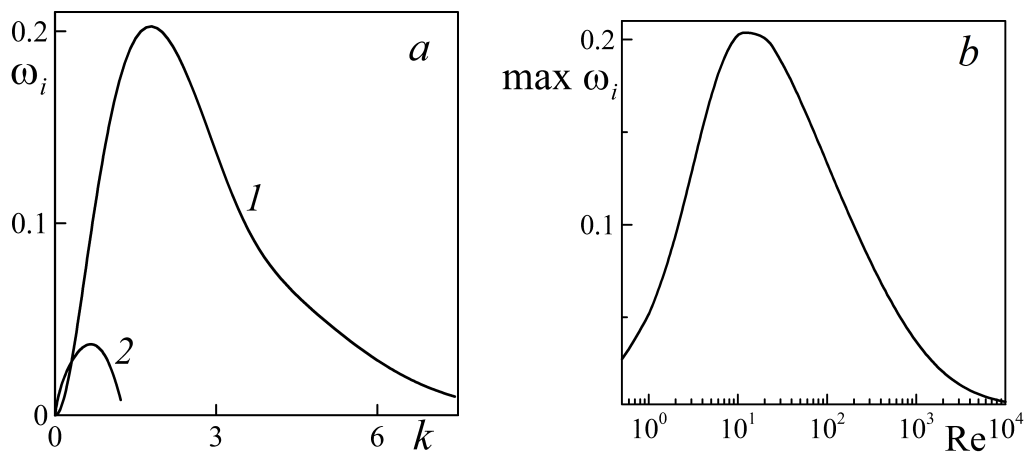

Figure 2. Wave amplification factor ω_i against the wave number k for $R = 10$ and 1000 (Curves *1* and *2* respectively) (*a*); Maximum wave amplification factor $\max \omega_i(k)$ against the Reynolds number Re (*b*). $C_1 = 0.001$, $C_2 = 0.1$, $y_0 = 0.5$, $\varepsilon = 0.1$, $\beta = 100$, $\eta = 1$

The dependence of instability on the particle inertia parameter β and the particle-to-fluid density ratio η is weaker. Given all other governing parameters fixed, the instability is strongest for particles with the particle inertia parameter of order unity (Fig. 3*a*). The magnitude of the particle-to-fluid density ratio η affects the flow only slightly (Fig. 3*b*). Varying η in the physically reasonable range results in varying the maximum increment of growing waves by less then 10%.

It is found that the instability strongly depends on the shape of the particle concentration profile in the main flow $C_0(y)$. A decrease in the magnitude of the particle concentration C_2 results in weakening the instability (Fig. 4*a*). In the case of "narrow" and "wide" particle concentration profiles ($y_0 < 0.02$ and $y_0 > 0.85$), the

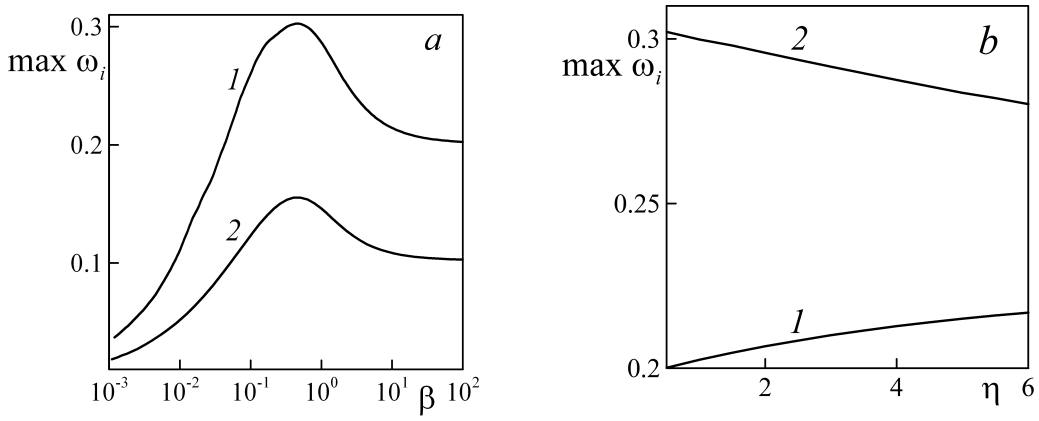

Figure 3. Maximum wave amplification factor $\max \omega_i(k)$ against the particle inertia parameter β for $C_2 = 0.1$ and 0.05, $\eta = 1$ (Curves *1* and *2* respectively) (*a*); maximum wave amplification factor $\max \omega_i(k)$ against the particle-to-fluid density ratio η for $\beta = 100$ and 0.33, $C_2 = 0.1$ (curves *1* and *2* respectively) (*b*). $C_1 = 0.01C_2$, $y_0 = 0.5$, $\varepsilon = 0.1$

instability triggered at low Reynolds numbers is damped (Fig. 4*b*). The wave growth increment is highest when the particles are distributed in the zone with the width of about 30% of the distance between the plates. In accordance with [1], the unstable Tollmien-Schlichting waves exist only for sufficiently high Reynolds numbers and particle mass concentrations higher than a certain threshold value (around 20%). Increase in the particle concentration gradient ε destabilizes the flow significantly.

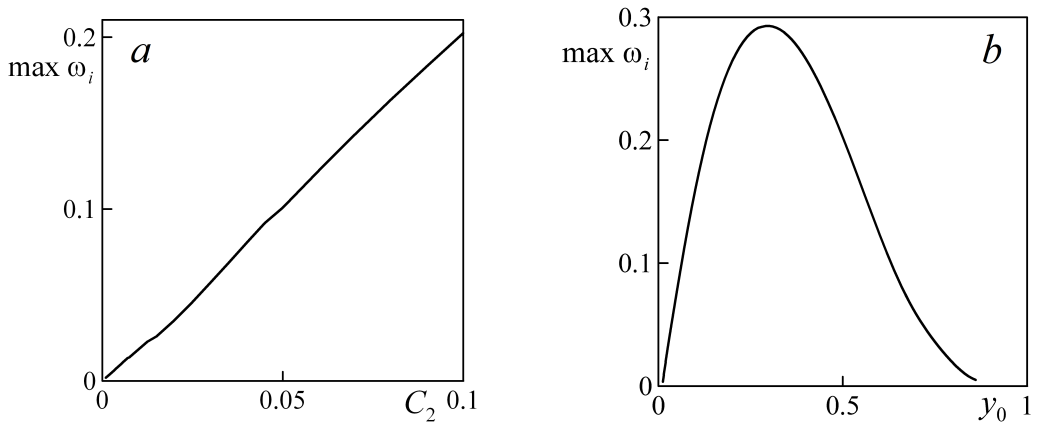

Figure 4. Maximum wave amplification factor $\max \omega_i(k)$ against the particle volume fraction in the core region C_2 for $y_0 = 0.5$ (*a*); maximum wave amplification factor $\max \omega_i(k)$ against the width of the particle distribution zone y_0 for $C_2 = 0.1$ (*b*). $C_1 = 0.01C_2$, $\varepsilon = 0.1$, $\beta = 100$, $\eta = 1$

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References

[1] Rudyak, V.Ya., Isakov, E.B., Bord, E.G., *Letters to ZhTF in Russian*, 24: 76–80 (1998).
[2] Marble, F.E., *Ann. Rev. Fluid Mech.*, 2: 397–446 (1970).
[3] Saffman, P.G., *J. Fluid Mech.*, 13: 120–128 (1962).
[4] Boronin, S.A., *Fluid Dynamics*, 43: 873–884 (2008).
[5] Brinkman, H.C. *Appl. Sci. Res. Sect. A.*, 1: 27–34 (1947).
[6] Einstein, A. *Ann. Phys.*, 19: 289–306 (1906).
[7] Godunov, S.K., *Uspekhi Mat. Nauk in Russian*, 16: 171–174 (1961).