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To the Theory of Drop Shattering in a Speedy Gas Flows

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Abstract

The general differential equations of shattering kinetics for mass efflux from fragmented in speedy gas flow drop and for quantity of stripped daughter droplets are derived on a base of mechanism of gradient instability in conjugated boundary layers on drop surface. At some assumptions the system is integrated and laws of parent drop mass diminishing, drop motion, as well as distribution function for stripped droplets by sizes, are obtained theoretically. The time and conditions for full drop breakup are found. Intermediate and final distributions of stripped droplets by sizes are calculated and discussed for various values of definitive parameters of the problem. Comparison of approximated results with those obtained by more precise numerical scheme get evidence of good enough agreement. Some general peculiarities of dispersion kinetics are described.

Introduction

The shattering of drops is a key process in many of phenomena. It sets a sizes, quantity and moments of tearing off of stripped daughter droplets. Just these parameters control over following motion and evaporation of a lot of finest droplets, which are gone with a gas flow into a wake of parent drop and form a two-phase combustible spray there. To describe quantitatively kinetics of further processes of stripped mass motion, evaporation and formation of combustible mixture one must first build a model of shattering process, which would be able to predict droplets sizes and moments of their tearing off. The core element of mathematical model of prepairing processes should be the distribution function of daughter droplets quantity by sizes, and its evolution in space and time is required. To determine kinetic regularities of shattering process experimentally is too complex problem, because the event picture is shadowed by a dense mist of huge number of finest droplets and their vapors. Hence, the regularities may be found by means of theoretical modelling of the process.

An investigation of instability of drop surface with due regard to velocity profile changing across conjugated boundary layers, as well as to changing of boundary layers thickness along drop surface [1] revealed for weak-viscosity liquids a new type of hydrodynamic instability – "gradient instability". As distinct from Kelvine – Helmholtz type, which is grounded on pressure difference action, mechanism of gradient instability consists in action of large gradient of inertia forces, caused by huge velocity gradient $10^5 \div 10^7\,\mathrm{sec}^{-1}$ inside perturbed liquid boundary layer. The inertia forces produce accelerations there as great as $10^3 \div 10^5 g$, which throw out finest liquid particles of radius, which is proportional to boundary layer thickness. This mechanism explains the "stripping" mode of shattering as high-frequency periodic dispergation from unstable part of drop surface and predicts the main features of event in speedy flows [2]. Its numerical application permitted to build the elementary theory of stationary detonation wave in aerosols [3]. The application may be done in less precise, but analytic form of approximative regularities for law of parent drop mass history (ablation law), law of shattering drop motion and distribution function of droplets stripped to any time moment, which is given below.

The Dispersion Mechanism

Flow around arbitrary ground Δl on drop surface is characterized by continuous velocity profile in conjugated boundary layers (fig. 1, φ – polar angle of a ground). Boundary layers thicknesses in liquid $\delta_l(\varphi,t)$ and gas $\delta_{\mathrm{g}}(\varphi,t)$ are expressed by gas flow velocity V_{g} and velocity on drop surface V_{s} as $\delta_l = Q2R_0/\mathrm{Re}_l^{0.5}$ and $\delta_{\mathrm{g}} = Q2R_0/\mathrm{Re}_{\mathrm{g}}^{0.5}$, where $\mathrm{Re}_l = \rho_l V_{\mathrm{s}} d_0/\mu_l$, $\mathrm{Re}_{\mathrm{g}} = \rho_{\mathrm{g}}(V_{\mathrm{g}} - V_{\mathrm{s}})2R_0/\mu_{\mathrm{g}}$ – Reynolds numbers for flows in liquid and gas boundary layers. Characteristic values of velocity gradients in boundary layers are bounded by condition of equality of viscous shear stresses on drop surface: $\mu_l V_{\mathrm{s}}/\delta_l = \mu_{\mathrm{g}}(V_{\mathrm{g}} - V_{\mathrm{s}})/\delta_g$. By eliminating Q, we obtain:

$$\frac{V_{\mathrm{s}}}{V_{\mathrm{g}}} = \frac{(\alpha\,\mu)^{1/3}}{1+(\alpha\,\mu)^{1/3}}, \qquad \frac{\delta_l}{\delta_{\mathrm{g}}} = \frac{\alpha}{(\alpha\,\mu)^{2/3}}, \tag{1}$$

where $\alpha \equiv \rho_{\mathrm{g}}/\rho_l$, $\mu \equiv \mu_{\mathrm{g}}/\mu_l$ are the ratios of gas and liquid densities and viscosities.

An investigation of instability of flow with continuous polygonal velocity profile had led in [1], [2] to a

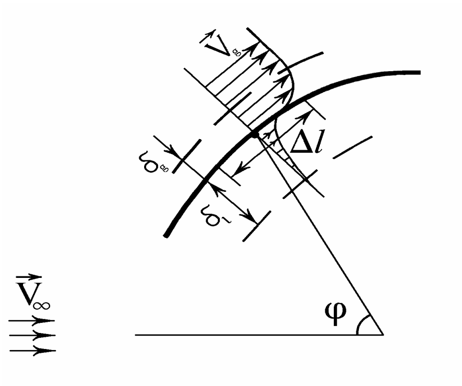

Figure 1 Scheme of flow around the elementary ground of drop surface.

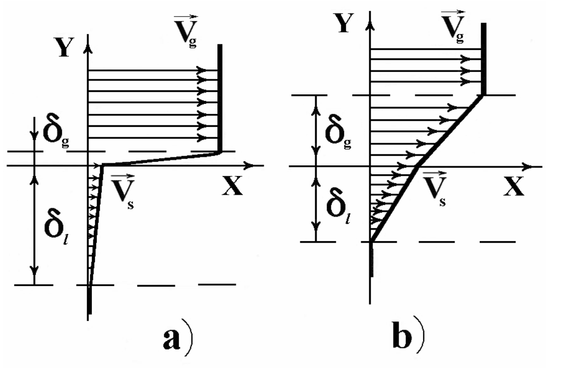

Figure 2 Continuous velocity profile, which was investigated on stability

conclusion that when $\nu_l > \nu_g$ the instability is determined by classic Kelvine – Helmholtz root, since $\delta_l \gg \delta_g$, $V_s \ll V_g$, and profile has a shape which is closed to tangential discontinuity (fig. 2,a) (ν is kinematic viscosity). But for most of liquid drops tested in experiments an inverse inequality takes place: $\nu_l < \nu_g$. Then boundary layers thicknesses are comparable: $\delta_l = O(\delta_g)$, and velocity profile in boundary layers becomes inflated (fig. 2, b). The unstable root ω of characteristic equation of boundary-value problem for disturbances in this case essentially differs from that of Kelvine – Helmholtz type. It defines another type of instability – gradient instability [1], [2], which has unstable mechanism working inside liquid boundary layer due to huge velocity gradient. As the main terms of characteristic equation contain none of gas parameters ρ_g, V_g, δ_g, gas flow doesn't influence on disturbances, but just transfers momentum to liquid by means of viscous shear stresses in undisturbed flow.

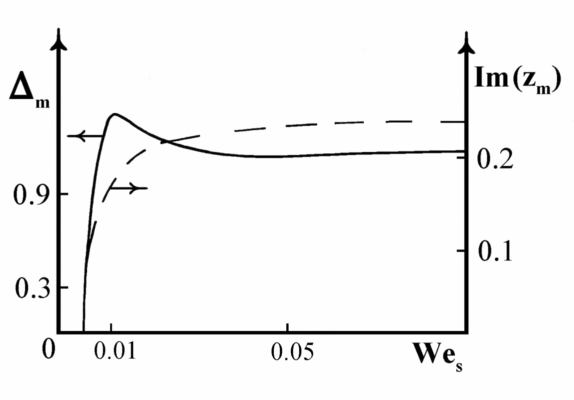

Figure 3 Dependencies $\Delta_m(\mathrm{We}_s)$, $\mathrm{Im}(z_m(\mathrm{We}_s))$

The dominant disturbance is of most interest in applications as it may realize the tearing-off of a liquid particle at non-linear stage. Dimensionless wavenumber $\Delta_m \equiv 2\pi\delta_l / \lambda_m$ and increment of dominant disturbance $\mathrm{Im}(z_m)$ for gradient instability depend only on "surface" Weber number $\mathrm{We}_s \equiv \rho_l V_s^2 \delta_l / \sigma$; these dependencies are given on fig. 3. They show that there exists critical value $\mathrm{We}_{s.cr.} = 0.004$ such that at $\mathrm{We}_s > \mathrm{We}_{s.cr.}$ the flow is unstable; approximately at $\mathrm{We}_s > 0.03$ the parameters become independent from We_s and equal to $\Delta_m = 1.225$, $\mathrm{Im}(z_m) = 0.25$. These regularities of dominant disturbance of gradient instability permit below to obtain differential equations for drop mass changing in time and for torn off droplets quantity.

Accounting for variation of flow parameters along drop surface, we find, that there exists a critical point $\varphi_{cr}(t)$ in which $\mathrm{We}_s(\varphi_{cr}) = 0.004$, and it divides drop surface on stable $\varphi < \varphi_{cr}$ and unstable $\varphi > \varphi_{cr}$ parts. Let's assume, that streamlining of spherical drop is potential, $V_g = 1.5 V_\infty \sin\varphi$; distribution of boundary layer thickness along surface let's take in Ranger's form [4]: $\delta_g(\varphi,t) = 2.2 R(t) \mathrm{Re}^{-0.5}(t) \Psi(\varphi)$, where $\Psi(\varphi) \equiv ((6\varphi - 4\sin 2\varphi + 0.5\sin 4\varphi)/\sin^5\varphi)^{0.5}$. Then, accounting for (1), the condition for gradient instability to exist on drop surface yields:

$$\frac{2.475\,\alpha}{\left(1+(\alpha\mu)^{1/3}\right)^2}\sqrt{\tilde{R}(\tau)\left(1-W(\tau)\right)^3}\,\sin^2\varphi\;\Psi(\varphi)\;\mathrm{GI} > K\,, \qquad 2)$$

here $\mathrm{GI} \equiv \mathrm{We}_\infty / \mathrm{Re}_\infty^{0.5}$ is gradient instability criterion, $\tilde{R} = R/R_0$, $W = w/V_\infty$ are dimensionless drop radius and velocity, R_0 – initial drop radius, $\tau = t/t_{ch}$, $t_{ch} = 2R_0/\sqrt{\alpha} V_\infty$ – characteristic time scale, V_∞ – velocity of gas flow. Equality in (2) at $K = 0.004$ defines the value of φ_{cr}; at $\alpha\mathrm{GI} > 5.6\cdot 10^{-4}$ we have $\varphi_{cr} < \pi/2$. This means, that **part of drop surface adjacent to edge is unstable, providing a possibility of dispersing**. The values of $\varphi_{\rm cr}$ are small enough in flows past shock waves: $\varphi_{\rm cr} \ll \pi$, so, **most part of drop surface generates a mist of droplets**. Condition GI>0.3 was first obtained empirically by Rabin et al. [5] as condition for stripping mode of breakup. It was grounded theoretically in [2] as criterion of existing of gradient instability on drop surface.

Equation for Stripped Droplets Quantity

Thus, each elementary ground Δl on unstable part of drop surface we can consider as a source of tiny daughter droplets (fig. 4). Its hydrodynamic mechanism is governed by regularities $\Delta_{\rm m}({\rm We_s})$, ${\rm Im}(z_{\rm m}({\rm We_s}))$, given on fig. 3. Then value of radius of torn droplets naturally to consider to be proportional to wavelength: $r(\varphi) = k_{\rm r}\lambda_{\rm m}(\varphi)$, $k_{\rm r} < 0.25$, while period of their tearing-off $t_{\rm i}$ – to time $t_{\rm m}$ of e-fold growth of disturbance's amplitude $t_{\rm m} = {\rm Im}^{-1}(z_{\rm m})\delta_l(\varphi)/V_{\rm s}(\varphi)$: $t_{\rm i}(\varphi) = k_{\rm t}t_{\rm m}(\varphi)$, $k_{\rm t} \approx 1$. The quantity of wavelengths, which are confined within a ground, is equal to $n(\varphi) = \Delta l(\varphi)/\lambda_{\rm m}(\varphi)$. Due to axial symmetry this is the quantity of torus of radius $R(t)\sin\varphi$, which were torn from spherical belt corresponding to Δl.

By relating volume of torus $\Delta\upsilon$ torn by time interval $t_{\rm i}$ to the stripped droplet volume, we obtain **equation for quantity of torn off droplets**:

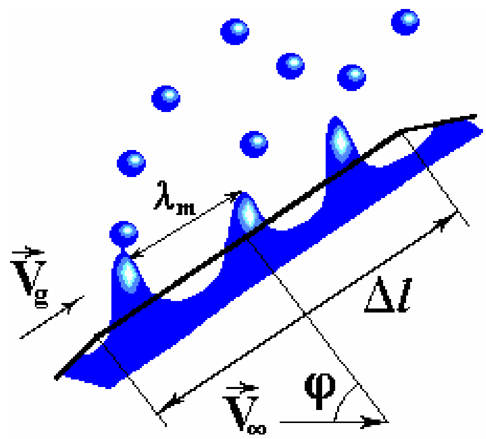

Figure 4 Scheme of dispersing on the elementary ground Δl of drop surface

$$\Delta n(\varphi,\tau) = \dot{n}'(\varphi,\tau) = B_2\sqrt{\tilde{R}(\tau)(1-W(\tau))^5}\,\frac{\sin^2\varphi}{\Psi^3(\varphi)}\Delta\varphi\Delta\tau\,, \tag{3}$$

$$\tilde{r}(\varphi,\tau) = B_1 T(\tau)\Psi(\varphi)\,, \qquad \tilde{r} = \frac{r}{R_0}\,, \qquad T(\tau) = \left(\frac{\tilde{R}(\tau)}{(1-W(\tau))}\right)^{0.5}, \tag{4}$$

where $B_1 = \dfrac{3.11\pi k_{\rm r}}{\Delta_{\rm m}({\rm We_s}){\rm Re}_\infty^{0.5}}\left(\dfrac{\alpha}{\mu^2}\right)^{1/3}$, $B_2 = \dfrac{0.30\Delta_{\rm m}^2({\rm We_s}){\rm Im}(z({\rm We_s})){\rm Re}_\infty^{1.5}}{\pi k_{\rm r}k_{\rm t}\left(1+(\alpha\mu)^{1/3}\right)}\left(\dfrac{\mu^2}{\alpha}\right)^{7/6}$. To obtain quantity of torn off daughter droplets differential equation (3) ought to be integrated in variables φ,τ.

Equation for Parent Drop Ablation

Since tearing-off occurs in time period $t_{\rm i}$, the rate of mass efflux can be expressed as:

$$\frac{\Delta m}{\Delta t}(\varphi,t) = \frac{\rho_l\Delta\upsilon(\varphi,t)}{t_{\rm i}(\varphi,t)} = \frac{\rho_l 2\pi^2k_{\rm r}^2R(t)\lambda_{\rm m}(\varphi,t)\sin\varphi\; V_{\rm п}(\varphi,t)\,{\rm Im}\left(z_{\rm m}({\rm We_s}(\varphi,t))\right)}{k_{\rm t}\delta_l(\varphi,t)}\Delta l(\varphi,t) \tag{5}$$

By substituting $\lambda_{\rm m} = 2\pi\delta_l/\Delta_{\rm m}$ in (5), taking into account $\Delta l = R(t)\Delta\varphi$, and by integrating over windward surface from $\varphi = \varphi_{\rm cr}(t)$ to drop edge $\varphi = \pi/2$, we obtain for the rate of mass diminishing:

$$\frac{dm}{dt} = -\frac{4\pi^3k_{\rm r}^2}{k_{\rm t}}\rho_l R^2(t)\int\limits_{\varphi_{\rm cr}(t)}^{\pi/2}\frac{V_{\rm s}(\varphi,t)\,{\rm Im}\left(z_{\rm m}({\rm We_s}(\varphi,t))\right)\sin\varphi}{\Delta_{\rm m}\left({\rm We_s}(\varphi,t)\right)}d\varphi \tag{6}$$

Function $F({\rm We_s}) \equiv {\rm Im}(z_{\rm m}({\rm We_s}))/\Delta_{\rm m}(z_{\rm m}({\rm We_s}))$, which defines in (6) an influence of parameters of dominant wave on a rate of mass efflux, is almost constant in the most part of diapason ${\rm We_s} > {\rm We_{s.cr.}}$ (fig. 5). In vicinity of critical point $F({\rm We_s})$ sharply decreases, so we can neglect by the rate of mass efflux at $0.004 < {\rm We_s} < {\rm We_s}(\varphi_1) \approx 0.006$. Besides, this vicinity must be avoid as wavelength of dominant wave here is greater than drop diameter, and its characteristic time interval is greater than drop breakup time $t_{\rm b}$ (fig. 3).

Therefore the corresponding disturbances do not able to terminate till breakup time, so, lower limit in (6) must be defined by such a value φ_1, for which $k_r\lambda_m(\varphi_1) \lessapprox R(t)$, $k_t t_m(\varphi_1) \lessapprox t_b$. Accounting this, we can put $F(\mathrm{We_s}) \approx 0.18$ within the interval $\varphi_1 < \varphi \le \pi/2$. With $V_s(\varphi,t) = 1.5(\alpha\,\mu)^{1/3}(1+(\alpha\,\mu)^{1/3})^{-1}(V_\infty - w(t))\sin\varphi$, we have

$$\frac{dm}{dt} = -\frac{6\pi^3 k_r^2 \rho_l}{k_t\left(1+(\alpha\,\mu)^{1/3}\right)}(\alpha\,\mu)^{1/3} R^2(t)\int_{\varphi_1(t)}^{\pi/2} 0.18\left(V_\infty - w(t)\right)\sin^2\varphi\, d\varphi. \qquad 7)$$

Calculating the integral, we obtain **the kinetic equation of drop shattering** [6]:

$$\frac{dm}{dt} = -\frac{1.08\pi^3 k_r^2 \rho_l}{k_t\left(1+(\alpha\,\mu)^{1/3}\right)}(\alpha\,\mu)^{1/3} R^2(t)\left(V_\infty - w(t)\right)\left(\frac{\pi}{4} - \frac{\varphi_1}{2} + \frac{\sin 2\varphi_1}{4}\right). \qquad 8)$$

Right-hand side of (8) expresses the dependence of mass efflux rate on all the main parameters of the process: current radius and velocity of drop, velocity of gas stream, physical properties of the media. Surface tension acts stabilizing through parameter φ_1 implicitly: as σ increases, values of $\mathrm{We_s}$ decrease on each ground, and critical point shifts to the drop edge, at the same time φ_1 increases and area of dispergation decreases.

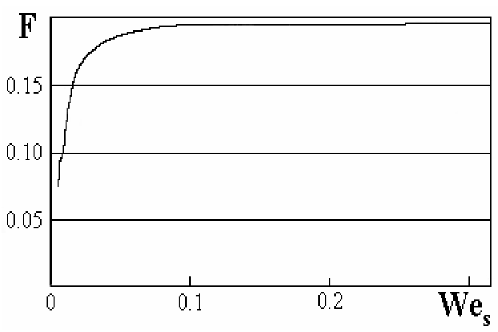

Figure 5 Dependence $F(\mathrm{We_s})$

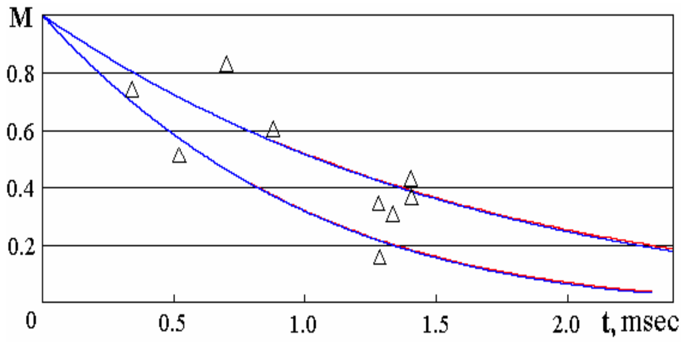

Figure 6 Dependencies $M(\tau)$; $\mathrm{We}_\infty = 1.18\cdot 10^3$; top – $k_r = 0.15$, bottom – $k_r = 0.19$, Δ – experiment [7]

Equation (8) may be rewritten in dimensionless form

$$\frac{dM}{d\tau} = -A\tilde{R}^2(\tau)\left(1 - W(\tau)\right)\left(1 - \frac{2\varphi_1(\tau)}{\pi} + \frac{\sin 2\varphi_1(\tau)}{\pi}\right), \qquad A \equiv \frac{0.405\pi^3 k_r^2}{k_t(1+(\alpha\,\mu)^{1/3})}\left(\frac{\mu^2}{\alpha}\right)^{1/6}, \qquad 9)$$

where A is character (initial) rate of mass efflux, $M = m/m_0$.

To determine $M(\tau)$ (9) requires simultaneous integration of equation of parent drop motion to determine $W(\tau)$ and eq. (2) at $K = 0.006$ – to determine $\varphi_1(\tau)$. But in the case $\varphi_1 \ll \pi$, $\sin 2\varphi_1 \approx 2\varphi_1$, and for spherical drop $M(\tau) = \tilde{R}^3(\tau)$ **we obtain the ablation law immediately**

$$M(\tau) = \left(1 - A\left(\tau - \alpha^{0.5} X_d(\tau)\right)/3\right)^3, \qquad \tilde{R}(\tau) = 1 - A\left(\tau - \alpha^{0.5} X_d(\tau)\right)/3, \qquad 10)$$

which indicates evidently the direct influence of law of parent drop motion $X_d = X_d(\tau)$ on its ablation law. Let's now use empirical data of Reinecke, Waldman [7] and write down law of drop motion in the form $\sqrt{\alpha}X_d(\tau) = \tau - (1-\exp(-H\tau))/H$, where $H = 2\sqrt{\alpha}$ is characteristic drop acceleration. This yields **the law of motion of shattering drop and its ablation law** [6]:

$$W = 1 - \exp(-H\tau), \qquad M = \tilde{R}^3 = \left(1 - h(1 - \exp(-H\tau))\right)^3. \qquad 11)$$

Parameter $h = A/3H$ reflects the relation between two governing factors of shattering: the rate of mass efflux ($\sim A$) and the rate of relaxational reducing of relative velocity of gas flow and parent drop ($\sim H$). As (11) shows, when $h > 1$, drop is completely shattered to time moment $\tau_b = H^{-1}\ln(h/(h-1))$; when $h < 1$ dispersion terminates before the whole drop breaks because latter factor leads to quick reducing of main reason of dispersion – relative velocity $1-W$, as eq. (9) shows. Remnant has radius $\tilde{R}_{rem} = 1-h$ and it may be shattered by another mechanism, for example, by Rayleigh – Taylor instability [2]. The analysis shows, that values of h for detonative systems are slightly higher then $h = 1$, values $h > 4$ correspond to ablation of liquid meteoroids and case $h < 1$ – to incomplete shattering of viscous drops. Comparison of ablation law (11) with experimental data [7] (fig. 6) indicates their good enough agreement [6].

Distribution Function in the Case When $h = 1$

To obtain distribution function $f_n(\tilde{r},\tau) = \Delta n(\tilde{r},\tau)/\Delta\tilde{r}$ we need to integrate (3) along each line $\tilde{r}(\varphi,\tau) = const$ inside strip $\Delta\tilde{r} = const$; sets of these lines are different for $h > 1$ and $h < 1$ as fig. 7 shows. Dispersing process begins in base range $\tilde{r}_{0l} < \tilde{r} < \tilde{r}_{0r}$, which corresponds to initial interval $\varphi_1(0) < \varphi < \pi/2$, and continuous in domain **A**. As equation (4) shows, the range of droplets sizes is then widened in domain **B** by fine fractions for $h > 1$ cases and coarse fractions for $h < 1$ cases.

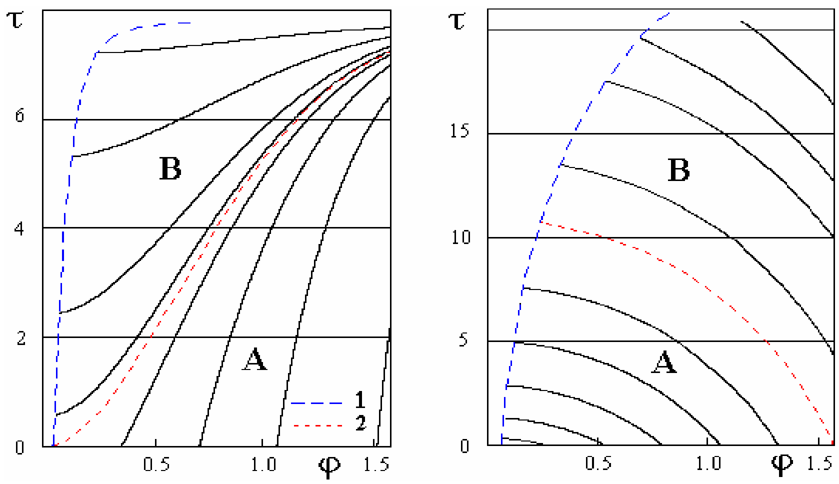

Figure 7 Set of lines $\tilde{r}(\varphi,\tau) = const$ (black, solid). Left: $h = 1.5$. Right: $h = 0.5$. ***1*** – left boundary $\varphi_l = \varphi_1(\tau)$ of domain of dispersing; ***2*** – border of domains **A** and **B**: $\tilde{r} = \tilde{r}_{0l}$ on the left and $\tilde{r} = \tilde{r}_{0r}$ on the right

In the case $h = 1$ eq. (11) gives $\tilde{R} = \exp(-H\tau) = 1 - W(\tau)$, then it follows from (4) that $T \equiv 1$, so, the lines $\tilde{r}(\varphi,\tau) = const$ are all parallel to time axis. *It appears in the remarkable case $h = 1$ of equality of ablation rate to rate of reducing of relative velocity, that for each fixed ground on drop surface the reducing of stripped droplets size due to reducing of parent drop size in time is strictly compensated by its growth due to reducing of relative velocity*. Thus, size of droplets, that torn off from fixed ground Δl, remains unchanged and it can be determined at $\tau = 0$. There is nothing to do now but to sum the quantity Δn in time. By eliminating $\Psi(\varphi)$ in (3) with a help of (4) and substituting $\Delta\varphi$ by $\Delta\varphi = \Delta\tilde{r}/B_1\Psi'$, where $\Psi'(\varphi) = \left(8 - 2.5\Psi^2(\varphi)\cos\varphi\right)/\Psi(\varphi)\sin\varphi$, we obtain **the distribution function $f_n(\tilde{r}) = \Delta n/\Delta\tilde{r}$ of daughter droplets, which were stripped to current moment τ** [6]:

$$\frac{\Delta n(\tilde{r},\tau)}{\Delta\tilde{r}} = f_n(\tilde{r},\tau) = \frac{1-\exp(-3H\tau)}{A\tilde{r}^2}\,\frac{B_1^3 B_2 \sin^3\varphi_0(\tilde{r})}{(8B_1^2 - 2.5\tilde{r}^2\cos\varphi_0(\tilde{r}))}, \quad for \quad \tilde{r}_{0l} < \tilde{r} < \tilde{r}_{0r} \qquad 12)$$

$\varphi_0(\tilde{r})$ being an inverse function to $\tilde{r} = B_1\Psi(\varphi)$ in (4), which is defined at $\tau = 0$. Calculated at $h = 1.00$ distributions $\Delta n(\tilde{r})$ and $\Delta M(\tilde{r})$ are given in fig. 8 in their evolution in time, which are self-similar in view of dependence (12), unlike general case $h \neq 1$.

The modal radius of distribution $\tilde{r}_{\mathrm{mod}}$ can be easily obtained from (12); it gives $\tilde{r}_{\mathrm{mod}} \approx 2B_1$. As well, the approximated formula for total quantity of stripped droplets $N = \sum_{\Delta r} \Delta n$ can be found in the form:

$$N \approx 0.047 B_2 A^{-1}\left(1.45 - 0.76\varphi_{l0} + 0.25\sin(3.05\varphi_{l0})\right). \qquad 13)$$

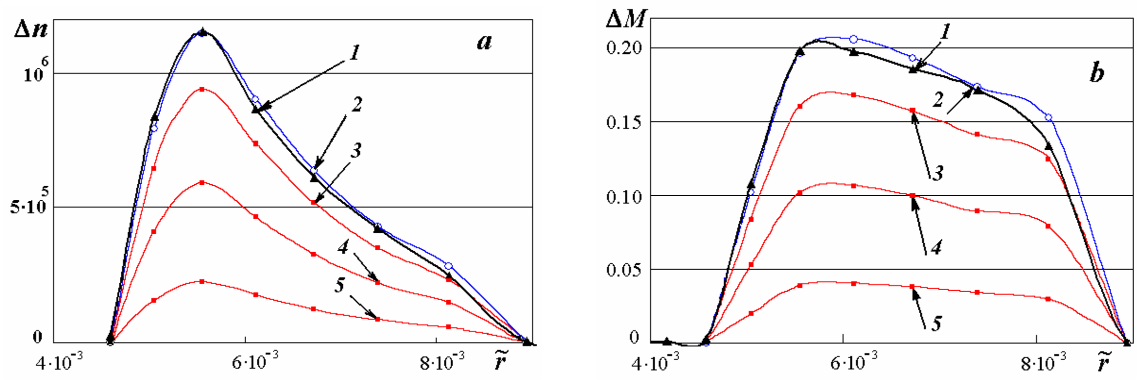

Figure 8 Distributions of stripped droplets by sizes: $\Delta n(\tilde{r})$ (left), and $\Delta M(\tilde{r})$ (right); $h = 1.04$, GI=49.4, $\mathrm{Re}_\infty = 1.3\cdot10^5$, $\tilde{r}_{0l} = 4.96\cdot10^{-3}$, $\tilde{r}_{0r} = 8.51\cdot10^{-3}$. Calculated by strict numerical scheme – ***1***; calculated by equation (12): ***2*** – final distribution at $\tau = 17.5$; ***3***, ***4***, ***5*** – intermediate distributions at $\tau = 3.5$, $\tau = 1.5$, $\tau = 0.45$

Distribution Function in General Case

After eliminating $\Psi(\varphi)$, substituting $\Delta\tau = \Delta\tilde{r}/(B_1\dot{T}(\tau)\Psi(\varphi))$ and integrating (3) along $\tilde{r}(\varphi,\tau) = const$ from $\varphi_* = \varphi_0 = \varphi(0)$ to $\varphi^* = \pi/2$ (see fig. 7), we obtain **the equation for distribution function**

$$\Delta n = f_{\mathrm{n}}\Delta\tilde{r} = 2\frac{B_1^3 B_2}{(h-1)H\tilde{r}^4}\int_{\varphi_0}^{\pi/2}\tilde{R}^3(\tau(\varphi))\left(1 - W(\tau(\varphi))\right)\sin^2\varphi\, d\varphi\;\Delta\tilde{r} \qquad 14)$$

where $\tau(\varphi)$ must be determined for each fixed $\tilde{r}$ from (4). Equation (14) directly points on the influence of ablation and motion laws of drop on the distribution function. To evaluate the curvilinear integral is a knotty problem in view of $\tau(\varphi)$ kind (see eq. (4) at $\tilde{r} = const$), so, one possible way is to approximate path of integration by straight line $\tau - \tau_* = (\varphi - \varphi_*)/a_{\mathrm{ef}}$ with some effective slope $a_{\mathrm{ef}}(\tilde{r},h)$. With this we obtain [8] from (14) using (11)

$$\Delta n(\tilde{r}) = f_{\mathrm{n}}(\tilde{r})\Delta\tilde{r} = \frac{B_1^3 B_2}{(h-1)\tilde{r}^4}\frac{a_{\mathrm{ef}}(\tilde{r})}{H^2}\sum_{i=1}^{4} A_i\left[\Phi_{i*}(\tilde{r}) - \Phi_i^*(\tilde{r})\right]\Delta\tilde{r} \qquad 15)$$

where $\Phi_i(\tilde{r}) = C^i(\tilde{r})\left(\sin^2\varphi(\tilde{r}) + \sin^2\left(\varphi(\tilde{r}) + \theta_i(\tilde{r})\right)\right)$ and $C(\tilde{r}) = (h-1)/\left(h - \left(\tilde{r}/(B_1\Psi(\varphi))\right)^2\right)$ must be calculated at lower $\varphi = \varphi_*(\tilde{r})$ and upper $\varphi = \varphi^*(\tilde{r})$ limits of integration; $\theta_i = \pi - \gamma_i$ at $h < 1$ and $\theta_i = \gamma_i$ at $h > 1$, $\gamma_i = \arcsin((iH/2a_{\mathrm{ef}})^2 + 1)^{-0.5}$, $A_i = 0.25 C_4^i h^{i-1}(1-h)^{4-i}$. Values of $\varphi_*(\tilde{r})$ and $\varphi^*(\tilde{r})$ are to be found from system of equations of line $\tilde{r}(\varphi,\tau) = const$ (4) and of dispersion region boundaries $\varphi_l = \varphi_1(\tau)$, $\varphi_r = \pi/2$ respectively (see fig. 7). Analysis of behavior of these lines permitted to find expression for a_{ef}, which is valid in wide diapason of h. As quantity of stripped droplets decreases with time (see (3)), effective slope must be fitted with account for most influence of its initial values and less influence of its mean values $a_{\mathrm{m.v.}} = (\varphi^* - \varphi_*)/(\tau^* - \tau_*)$. Besides, it is necessary to set natural demand to get in the limit $h \to 1$ the exact expression (12), obtained in case $h = 1$. Eventually we came to following expressions for a_{ef} in domains **A** and **B**:

$$a_{\text{ef A}} = \frac{\left(h-1+h^{-2}+k(h)|h-1|^{0.5}h^{-1}\right)a_{\text{m.v.}}a_*}{(h-1)a_*+h^{-2}a_{\text{m.v.}}+(a_{\text{m.v.}}+a_*)|h-1|^{0.5}h^{-1}}, \qquad a_{\text{ef B}} = \frac{\left(k_1h+k_2h^{-3}\right)a_{\text{m.v.}}a_*}{ha_*+h^{-3}a_{\text{m.v.}}} \tag{16}$$

with $k(h)=0.93(2h-1)h^{-1}$ for $h>1$, $k(h)=1.33h^{-0.5}$ for $h<1$; $k_1=0.80$, $k_2=1.13$ for $h<1$, and $k_1=1.13$, $k_2=0.87$ for $h>1$. The calculated by formulae (12), (15)–(16) distributions are shown in figs. 8–10.

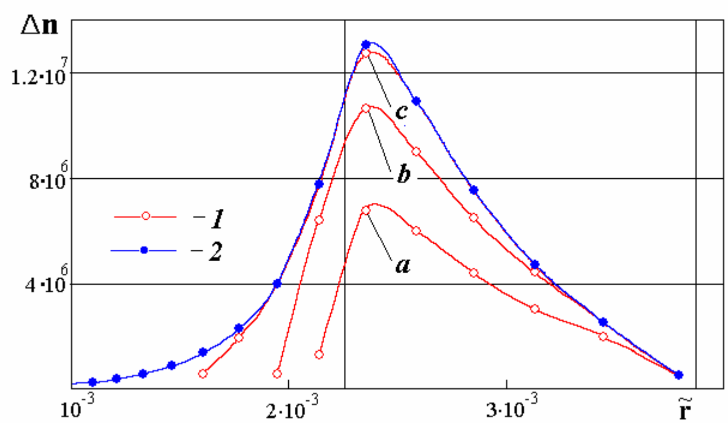

Figure 9 Calculated by formulae (12), (15)–(16) distributions of stripped droplets quantity. ***1*** : at $\tau=1.0$ – ***a***, $\tau=2.0$ – ***b***, $\tau=3.5$ – ***c***; ***2***: final distribution at $\tau=\tau_{\text{b}}=5.48$. Vertical lines – boundaries of the base range $\tilde{r}_{0l}=2.26\cdot10^{-3}$, $\tilde{r}_{0r}=3.87\cdot10^{-3}$; $h=2.0$.

Peculiarities of Droplets Distributions

The most part of mass and quantity of daughter droplets at any h are generated in a base range $\tilde{r}_{0l}<\tilde{r}<\tilde{r}_{0r}$ (domain **A** , fig. 7). At $h=1$ function (12) is self-similar, therefore range of sizes of stripped droplets $\tilde{r}_{\min}<\tilde{r}<\tilde{r}_{\max}$ does not vary in time and coincides with a base range $\tilde{r}_{0l}=\sqrt{3.2}<\tilde{r}<\tilde{r}_{0r}=\sqrt{3\pi}$. Since that, the interval of droplets sizes has the least width at $h=1$, and values of the mean diameters d_{ij} of distribution $f_{\text{n}}(\tilde{r})$ are close each other. In general case the basic range sets up only initial distribution, and then, as shattering proceeds, the bounds of distribution shift decreasing at $h>1$ and increasing at $h<1$ (fig. 7).

At $h>1$ function $\Delta n(\tilde{r})$ has ascending and descending branches, which form maximum at $\tilde{r}_{\text{mod}}$. It appears due to a smaller rate of production of droplets to the left of $\tilde{r}_{\text{mod}}$, while to the right – smaller is the period of existence of conditions for such a production (fig. 7). The shape of curve $\Delta n(\tilde{r})$ depends on values of h , as illustrated by fig. 10, while bench-mark values of $\tilde{r}_{0l},\tilde{r}_{0r},\tilde{r}_{\text{mod}},\Delta n(\tilde{r}_{\text{mod}})$ are defined by values of B_1,B_2 .

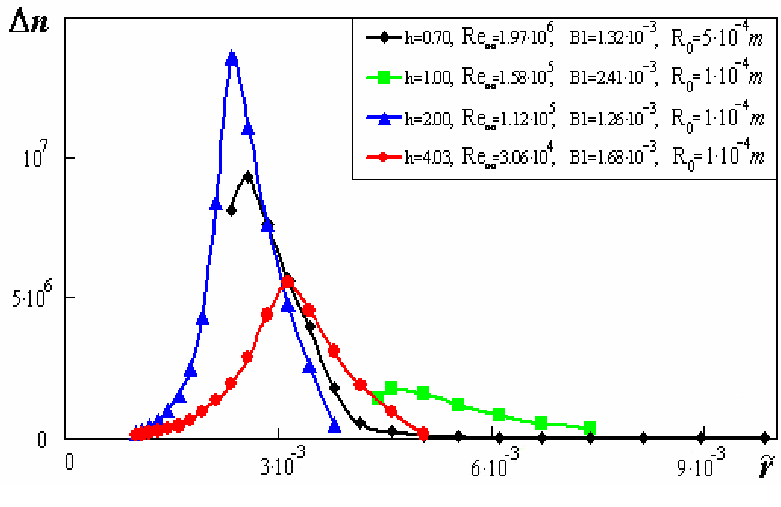

Figure 10 Distributions $\Delta n(\tilde{r})$ at various h, Re_∞

In accordance with (3), (4), sizes of the totality of daughter droplets are defined by parameter $B_1\sim\alpha^{1/3}\mu^{-2/3}\text{Re}_\infty^{-0.5}$, which plays the role of scale of sizes; as well, $B_2\sim\alpha^{-7/6}\mu^{7/3}\text{Re}_\infty^{1.5}$ is responsible for quantity scale. When $h\sim\alpha^{-2/3}\mu^{1/3}$ increases, the part of fine fractions widens and at $h\cong2$ it becomes comparable with that of coarse one (fig. 10).

Distribution Function for Theoretical Law of Drop Motion

Distribution functions (12), (15) were derived on a base of empirical law of drop motion (11), well enough fitted quantitatively. To eliminate arbitrary influence of the law's form, the procedure of obtaining $f_{\text{n}}(\tilde{r})$ was undertaken in the similar way, but it was grounded now on simultaneous integration of ablation equation (9) and differential equation of drop motion

$$\frac{dW}{d\tau} = C\frac{(1-W)^2}{\tilde{R}}, \tag{17}$$

where $C = 0.75\sqrt{\alpha}\,C_\mathrm{d}$, C_d being drop drag coefficient. By eliminating velocity, we obtain equation of second order with respect to radius: $\ddot{\tilde{R}} = \dot{\tilde{R}}^2 / h\tilde{R}$, dot means differentiating with respect to time. Solution of this equation with initial conditions $\tilde{R}(0) = 1$, $\dot{\tilde{R}}(0) = -A/3$ at $h = 1$ has exponential form: $\tilde{R} = \exp(-A\tau/3)$. From (17) it then follows: $1 - W = \exp(-C\tau)$. Therefore, at $h = 1$ *the laws of drop motion and ablation, and thus – distribution function, – coincide with those, obtained by use of empirical relaxational law of drop motion.*

In general case $h \neq 1$ the integration of system (9), (17) with initial values $\tilde{R}(0) = 1, W(0) = 0$ leads to power functions:

$$\tilde{R}(\tau) = \left(1 - C(h-1)\tau\right)^{h/(h-1)}, \qquad W(\tau) = 1 - \left(1 - C(h-1)\tau\right)^{1/(h-1)}. \tag{18}$$

By substituting (18) in equation (14) for distribution function and integrating again along straight $\tau - \tau_* = (\varphi - \varphi_*)/a_\mathrm{ef}$, which approximates path of integration, we obtain for natural $\eta = 3h/(h-1)$:

$$f_\mathrm{n}(\tilde{r}) = \frac{3hB_1^3 B_2}{(h-1)A\tilde{r}^4}\left[\frac{1}{c(\eta+1)}P_{\eta+1}(\varphi) - F_\eta(\varphi)\right]_{\varphi_*}^{\varphi^*} \tag{19}$$

where $F_\eta(\varphi) = \frac{\sin 2\varphi}{2}\sum_{k=0}^{\mathrm{E}(\eta/2)}(-0.25)^k P_\eta^{(2k)}(\varphi) + \frac{\cos 2\varphi}{4}\sum_{k=1}^{\mathrm{E}((\eta+1)/2)}(-0.25)^{k-1} P_\eta^{(2k-1)}(\varphi)$, $P_\eta(\varphi) = (b + c\varphi)^\eta$, $P_\eta^{(k)}$ – its k-th derivative, $b = 1 - (h-1)C(\tau_* - \varphi_*/a_\mathrm{ef})$, $c = -(h-1)C/a_\mathrm{ef}$. To natural $\eta > 3$ corresponds series of discrete values of h: $1 < h = \eta/(\eta-3) \le 4$. For integer $\eta < 0$ we obtain series of h values, which belongs to interval $0.25 \le h < 1$ of incomplete regimes of shattering; in this case f_n expresses in $\mathrm{Si}(\varphi)$, $\mathrm{Ci}(\varphi)$. Named set of η values covers compactly enough all the practically important diapason of h.

Distributions were calculated for $h = 1.5$; $h = 2.0$; $h = 4.0$ by the formulae (18)–(19). Comparison showed [9] weak influence on $\Delta n(\tilde{r})$ of the two methods – empirical and theoretical – of determination of drop motion law. The lack of base of empirical data about law of drop motion $x_\mathrm{d}(t)$ gives preference to formulae (18)–(19), which may be used for arbitrary gas – droplet systems.

Conclusions

Model of drop shattering which is based on mechanism of action of gradient instability provides elementary theory of drop breakup in speedy gas flows in the form of the main analytical relations. The ablation law and the distribution function of the daughter droplets stripped to a current moment are obtained at some simple assumptions. These relationships make possible to describe quantitatively further aerodynamics of accelerating and evaporating mist of stripped droplets, which develops inflammable mixture in a wake of a shattering drop, and may serve therefore as a ground for model of heterogeneous combustion [10].

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